

Friday, September 5, Lecture 4: Power Series II

Last time...

- Power and Taylor series about a :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n$$

- Taylor's theorem with remainder:

$$\underbrace{|P_n(x) - f(x)|}_{\substack{\text{error in } n^{\text{th}} \\ \text{Taylor approx}}} \leq M_{n+1} \frac{(x-a)^{n+1}}{(n+1)!}$$

- Solving (some) differential equations.

Today: Radius of convergence, asymptotic notation,
Dating Applications of Taylor series)

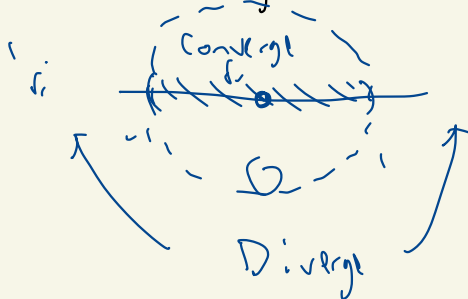
Taylor series are only valid in radius of convergence.

Ex: Geometric series

$$\sum_{k=0}^{\infty} x^k$$

converges if $|x| < 1$,

diverges otherwise



Finding Radii of convergence; compare to geom. series;

Ex:

Find Radius of convergence for

$$\sum_{n=0}^{\infty} (3x-1)^n$$

Soln:

This isn't quite a power series, but it isn't hard to adjust to one:

By geom. series, this converges if

$$|3x-1| = |r| < 1 \Leftrightarrow -1 < 3x-1 < 1$$

$$\Leftrightarrow 0 < 3x < 2$$

$$\Leftrightarrow 0 < x < \frac{2}{3}$$

$$\left(\begin{array}{ccc} & 1 & \\ 0 & 1/3 & 2/3 \end{array} \right)$$

$$\text{Interval of convergence} = (0, \frac{2}{3})$$

$$\text{Radius of convergence } R = 1/3 \text{ with center } x = \frac{1}{3}.$$



Asymptotic Notation

Common to say

$$\sin x = x - \frac{x^3}{3!} + \text{"lower order terms"}$$

But what does this mean? Well, by Taylor's theorem

$$\text{"lower order terms"} = |R_k(x)| \leq M_{k+1} \frac{x^{k+1}}{(k+1)!}$$

(here, $k=5$)

Notice:

Thanks to x^{k+1} , the error goes to 0 faster than any lower order polynomial. eg. for $k=3$,

$$\lim_{x \rightarrow 0} \frac{|R_3(x)|}{x} \leq C_4 \lim_{x \rightarrow 0} \frac{x^4}{x} = 0 \leftarrow \text{so } |R_3(x)| \rightarrow 0 \text{ faster than } x \rightarrow 0$$

$$\text{and similarly, } \lim_{x \rightarrow 0} \frac{x^4}{x^2} = 0 \leftarrow \text{and } |R_3(x)| \rightarrow 0$$

$$\text{and } \lim_{x \rightarrow 0} \frac{x^4}{x^3} = 0 \quad \checkmark \text{ faster than } x^2 \text{ or } x^3, \text{ (but not } x^4)$$

Def: Little o notation, big O notation

Let $f(x)$ and $g(x)$ be continuous functions.

We say $f(x) = o(g(x))$ as $x \rightarrow a$ if

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0$$

and we say $f(x) = O(g(x))$ as $x \rightarrow a$ if

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = C < \infty.$$

Ex:

$$x^2 \text{ is } o(x) \text{ as } x \rightarrow 0, \text{ since } \lim_{x \rightarrow 0} \frac{x^2}{x} = 0$$

$$\log x \text{ is } o(x) \text{ as } x \rightarrow \infty, \text{ since } \lim_{x \rightarrow \infty} \frac{\log x}{x} = 0$$

HW1!

$$\frac{x+a}{x^{2b+1}-c}, b>1 \text{ is } O\left(\frac{1}{x^{2b}}\right) \text{ as } x \rightarrow \infty, \text{ since}$$
$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x+a}{x^{2b+1}-c}\right)}{\left(\frac{1}{x^{2b}}\right)} = 1 < \infty.$$

This allows us to rigorously say "low order terms".

Ex:

$$\cos x = 1 - \frac{x^2}{2!} + o(x^2) \quad \text{as } x \rightarrow 0$$

Pf:

$$\underbrace{\cos x - \left(1 - \frac{x^2}{2!}\right)}_{R_2(x)} = \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\lim_{x \rightarrow 0} \frac{\frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{x^2} = \lim_{x \rightarrow 0} \frac{x^2}{4!} - \frac{x^4}{6!} + \dots = 0$$

Very useful in calculations when we want to say 

$$\sin x \approx x \quad \text{near } 0$$

correct statement:

$$\sin x = x + o(x) \quad \text{as } x \rightarrow 0.$$

Manipulation Rules:

① If $c \in \mathbb{R}$ and $f(x) = o(g(x))$, then $cf(x) = o(g(x))$

② If $f_1(x) = o(g_1(x))$ and $f_2(x) = o(g_2(x))$, then
 $f_1(x)f_2(x) = o(g_1(x)g_2(x))$

③ If $f(x) = o(g(x))$, then $\pi f(x) = o(\pi g(x))$

④ If $\lim_{x \rightarrow 0} g(x) = 0$, then $\frac{1}{1+g(x)} = 1 - g(x) + o(g(x))$

⑤ $o(f(x)) + o(g(x)) = o(f(x) + g(x))$

⑥ $o(o(f(x))) = o(f(x))$.

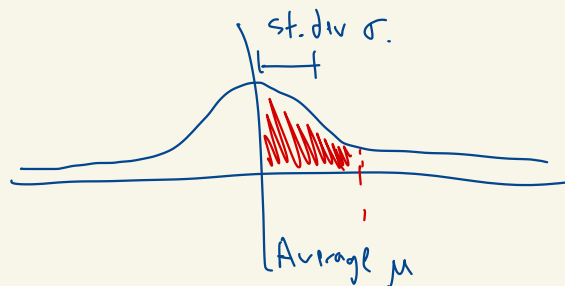
Ex:

$$\frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2} + o(x^2)} = 1 + \frac{x^2}{2} + o(x^2)$$

Way easier than computing Taylor series.

Application: Normal distribution.

Very common for random variables (like test scores or heights) to follow a "bell curve" or normal distribution:



Probability density function

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$\therefore \sigma=1, \mu=0 \Rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

want to know what fraction of American men are between 5'9" and 6'?

$$\mu = 5'9"$$

$$\sigma = 3"$$

$$\Rightarrow \text{Prob}(\text{Height between } 5'9" \text{ and } 6') = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$



This is a nasty integral. But we can estimate:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{(-\frac{1}{2}x^2)}{1!} + \frac{(\frac{-1}{2}x^2)^2}{2!} + o((-\frac{1}{2}x^2)^2) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4) \right)$$

$$\text{So } \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \approx \int_0^1 \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{6} + \frac{1}{40} \right)$$

$$\approx .342$$

Since $1/2$ people are shorter than average,

$$\text{Prob}(\text{Height} \leq 6') \approx .5 + .34 = .842$$

(for reference, a more accurate estimate is

$$\text{Prob}(\text{Height} \leq 6') \approx .8413447.$$

our estimate was really good!)

Moral: Math says not all dating bios are truthful!